Analytical model of SiPM time resolution and order statistics with crosstalk

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Time resolution models and approaches

Evergreen legacy classics

R. Post and L. Schiff, "Statistical Limitations on the Resolving Time of a Scintillation Counter," Phys. Rev., vol. 80, no. 6, pp. 1113–1113, Dec. 1950.

L. G. Hyman, "Time Resolution of Photomultiplier Systems," Rev. Sci. Instrum., vol. 36, no. 2, p. 193, 1965.

Modern approaches accounting for SiPM specifics

Monte Carlo simulations

W. W. Moses, W. S. Choong, and S. E. Derenzo Fundamental Limits of Timing Resolution for Scintillation Detectors *Lawrence Berkeley National Laboratory,* March 13, 2013

Classics : detection event statistics & SER IRF(t)

S. Seifert, H. T. Van Dam, R. Vinke, P. Dendooven, H. Löhner, F. J. Beekman, S. Member, and D. R. Schaart, "A Comprehensive Model to Predict the Timing Resolution of SiPM-Based Scintillation Detectors : Theory and Experimental Validation," IEEE TNS vol. 59, no. 1, pp. 190–204, 2012.

Order statistics of PE detection times T(n)

M. W. Fishburn and E. Charbon, "System Tradeoffs in Gamma-Ray Detection Utilizing SPAD Arrays and Scintillators," vol. 57, no. 5, pp. 2549–2557, 2010.

Cramer-Rao lower bound estimation

S. Seifert, H. van Dam, and D. Schaart, "The lower bound on the timing resolution of scintillation detectors," *Phys. Med.*, vol. 57, no. 7, pp. 1797–814, Apr. 2012.

MOTIVATION

- **Is there a chance for an analytical model of reasonable simplicity?**
  - **Utilizing ENF model of Photon Number Resolution**
    - **S. Vinogradov et al., IEEE TNS, 2011.**
  - **Accounting for crosstalk (CT number & time distributions)**
    - **S. Vinogradov, NIMA, 2011.**

- **Which way is more efficient and provides best results?**
  - **Classic: Independent Identically Distributed T(k:n) $\otimes$ IRF(t)**
  - **Order Statistics: Sorted & Dependent T(n)**

- **How it corresponds to Cramer-Rao Lower Bound?**

Naïve consideration based on total ENF

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$$\sigma_t = \frac{\sqrt{\sigma_{out}^{2} + \sigma_e^{2}}}{\frac{d\mu_{out}}{dt}} \qquad \sigma_{out}: \qquad \frac{\sigma_{out}^{2}}{\mu_{out}^{2}} = PNR_{out}^{2} \approx \frac{\sigma_{in}^{2}}{\mu_{in}^{2}} \cdot ENF_{total}\Big|_{in=Poisson} = \frac{1}{N_{ph} \cdot PDE} \cdot ENF_{total}\left(N_{ph}\right)$$

$$ENF_{total}\left(N_{ph}\right) \approx ENF_{gain} \cdot ENF_{ct} \cdot ENF_{dcr}\left(N_{dark}/N_{pe}\right) \cdot ENF_{binom}\left(N_{pe}/N_{cell}\right)$$

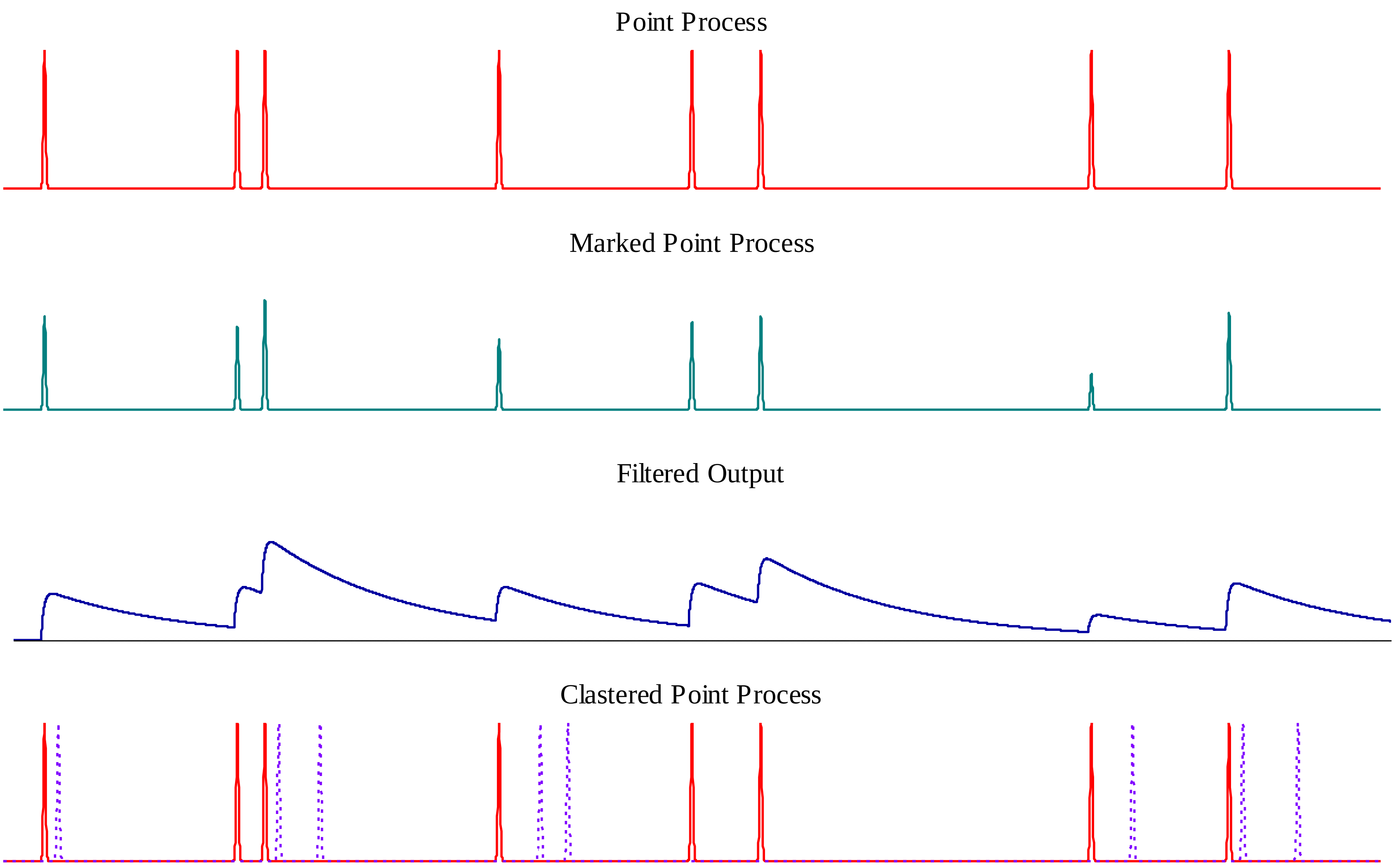
$$\mu_{out}(t) = \left\{\left[N_{ph}(t) * PDF_{SPTR}(t)\right] * PDF_{CT}(t)\right\} * IRF_{SER}(t)$$

$$\_\_\_\_\_\_\_\_\underline{primaries}\_\_\_\_\_\underline{crosstalk}\_\_\_\underline{instrumental\ response}$$

$$\sigma_t = \frac{\mu_{out}(t)}{\frac{d\mu_{out}(t)}{dt}} \cdot PNR_{sipm}\left(N_{ph}\right) = \frac{\sqrt{ENF_{total}\left(N_{ph}\right) + \frac{\sigma_e^{2}}{\mu_{out}(t)^{2}}}}{\frac{d\ln\left(\mu_{out}(t)\right)}{dt} \cdot \sqrt{N_{ph} \cdot PDE}} \quad \sim \quad \frac{Excess\ noise}{Order\ statistics?}???$$

$$Ideal\ time\ responses: \Big|_{PDF_{xxx}=\delta(t)} \qquad \sigma_t = \frac{\sqrt{N_{pe}(t)}}{I_{pe}(t)} \cdot \sqrt{ENF}\ \Bigg|_{Npe(t)=n} = \sqrt{n} \cdot \tau_{pe} \cdot \sqrt{ENF}\ \Bigg|_{ENF=1} = Erlang$$

Classical approach



Classical approach

- **Filtered output of Marked Poisson Point Process:**

$$X_{in}(t)=\sum_{i=1}^{N}\delta(t-t_i) \qquad N-Poissonian \qquad t_i-iidrv(i=1...N)$$

$$Y_{out}(t)=\sum_{i=1}^{N}IRF(t-t_i) \qquad IRF(t)=A_i\cdot h(t) \qquad A_i-iidrv(i=1...N)$$

$$E[Y_{out}(t)]=E[X_{in}(t)]*E[IRF(t)]=\overline{A}\cdot[\lambda*h](t) \qquad N=\int_0^t\lambda(t')dt'$$

$$Var[Y_{out}(t)]=COV[X_{in}]*COV[IRF](t-t')\Big|_{t'=t}=\overline{A}^2\cdot\left(1+\frac{\sigma_A^2}{\overline{A}^2}\right)\cdot[\lambda*h^2](t)$$

$$REMARK: COV[X_{in}]=\lambda(t)\cdot\delta(t-t') \qquad Var[X_{in}(t)]\underset{t'\to t}{\to}\infty$$

$$SIPM \quad specific$$

$$E[Vout(t)]=\overline{V}_{ser}\cdot[\lambda*h](t) \qquad \overline{V}_{ser}\approx\frac{q\cdot\overline{Gain}\cdot R_{load}}{\tau_{fall}} \qquad \lambda(t)=N_{ph}\cdot PDE\cdot[\rho_{ph}*\rho_{sptr}](t)$$

$$Var[V_{out}(t)]=\overline{V}_{ser}^2\cdot ENF_{gain}\cdot[\lambda*h^2](t)+V_{noise}^2$$

$$\sigma_t=\frac{\sqrt{Var[V_{out}(t)]}}{\dfrac{d\overline{Vout(t)}}{dt}}=\frac{\sqrt{N_{pe}\cdot ENF_{gain}\cdot[\rho_{ph}*\rho_{sptr}*h_{ser}^2](t)+\dfrac{V_{noise}^2}{\overline{V}_{ser}^2}}}{N_{pe}\cdot\dfrac{d[\rho_{ph}*\rho_{sptr}*h_{ser}](t)}{dt}}$$

Advanced classical approach

- **Non-Poissonian light statistics and "cascades" of secondary events (random amplification of primaries)**

H. H. Barrett, "Correlated point processes in radiological imaging," SPIE, vol. 3032, 1997.

J. Yao and I.A. Cunningham, "Parallel cascades: New ways to describe noise transfer in medical imaging systems", Medical Physics 28, 2001.

These papers derived results on mean and covariance of the point process only

$$X_{in}(t) = \sum_{i=1}^{N} \sum_{j}^{K_i} \delta(t - t_i - \Delta t_{ij}) \quad N - NonPoissonian: \quad (\overline{N}, Var[N] = \overline{N} + \delta_{sci}^2 \cdot \overline{N}^2)$$

$$Secondaries: \quad K_i - iidrv(i = 1...\infty) : (\overline{K}, Var[K]) \quad \Delta t_{ij} - iidrv(i = 1...N; j = 1...K_i) : (\rho_{\sec}(\Delta t))$$

$$\overline{V_{out}(t)} = \overline{V}_{ser} \cdot \overline{K}_{\sec} \cdot [\lambda * h](t) = \overline{N}_{pe} \cdot \overline{V}_{ser} \cdot \overline{K}_{\sec} \cdot [\rho_{ph} * \rho_{sptr} * \rho_{\sec} * h_{ser}](t)$$

$$Var[V_{out}(t)] = \overline{V}_{ser}^2 \cdot ENF_{gain} \cdot \overline{K}_{\sec}^2 \cdot ENF_{\sec} \cdot \{[\lambda * h^2](t) + \delta_{sci}^2 \cdot [\lambda * h]^2(t)\} + V_{noise}^2$$

$$\sigma_t = \frac{\sqrt{\overline{N}_{pe} \cdot ENF_{gain} \cdot ENF_{\sec} \cdot \{[\rho_{ph} * \rho_{sptr} * \rho_{\sec} * h_{ser}^2](t) + \delta_{sci}^2 \cdot \overline{N}_{pe}^2 \cdot [\rho_{ph} * \rho_{sptr} * \rho_{\sec} * h_{ser}]^2(t)\} + \dfrac{V_{noise}^2}{\overline{V}_{ser}^2 \cdot \overline{K}_{\sec}^2}}}{\overline{N}_{pe} \cdot \dfrac{d[\rho_{ph} * \rho_{sptr} * \rho_{\sec} * h_{ser}](t)}{dt}}$$

Applicability of ENF approach

- **It make sense in case of fast SER relative to detected events time distribution (mainly light pulse width)**

$$\sigma_t = \frac{\sqrt{\overline{N}_{pe} \cdot ENF_{gain} \cdot ENF_{\sec} \cdot \left\{ \left[\rho_{ph} * \rho_{sptr} * \rho_{\sec} * h_{ser}^2\right](t) + \delta_{sci}^2 \cdot \overline{N}_{pe}^2 \cdot \left[\rho_{ph} * \rho_{sptr} * \rho_{\sec} * h_{ser}\right]^2(t) \right\} + \frac{V_{noise}^2}{\overline{V}_{ser}^2 \cdot \overline{K}_{\sec}^2}}}{\overline{N}_{pe} \cdot \frac{d\left[\rho_{ph} * \rho_{sptr} * \rho_{\sec} * h_{ser}\right](t)}{dt}}$$

$$= \frac{\left[\rho_{ph} * \rho_{sptr} * \rho_{\sec} * h_{ser}\right](t)}{\frac{d\left[\rho_{ph} * \rho_{sptr} * \rho_{\sec} * h_{ser}\right](t)}{dt}} \cdot \sqrt{\frac{1}{\overline{N}_{pe}} \cdot ENF_{gain} \cdot ENF_{\sec} \cdot \left\{ \frac{\left[\rho_{ph} * \rho_{sptr} * \rho_{\sec} * h_{ser}^2\right](t)}{\left[\rho_{ph} * \rho_{sptr} * \rho_{\sec} * h_{ser}\right](t)} + \delta_{sci}^2 \right\} + \frac{V_{noise}^2}{\overline{V_{out}(t)}^2}}$$

$$if \quad SER\ is\ fast: \quad \left[f_{\det} * h_{ser}\right](t) \approx f_{\det}(t) \quad \Rightarrow$$

$$\sigma_t = \frac{1}{\frac{d \ln\left[\rho_{ph} * \rho_{sptr} * \rho_{\sec}\right](t)}{dt}} \cdot \sqrt{\frac{1}{\overline{N}_{pe}} \cdot ENF_{gain} \cdot ENF_{\sec} \cdot ENF_{sci} + \frac{V_{noise}^2}{\overline{V_{out}(t)}^2 = Thresh^2}}$$

Order Statistics approach

- Sorting iidrv Tn to form ordered set T1≤T2 ≤…Tn
- Decomposing PDF f(t) of Ti into PDF of T(i):

$$f_{k:n}(t) = n \cdot \binom{n-1}{k-1} \cdot f(t) \cdot F(t)^{k-1} \cdot (1-F(t))^{n-k} \qquad f(t) = [\rho_{ph} * \rho_{sptr}](t)$$

$$f_{k:N_{pe}}(t) \cong N_{pe} \cdot f(t) \cdot Gamma(N_{pe} \cdot F(t), k)$$

  - ***Remark: n is non-random, but could be approx. ~ Gamma distrib.***

- ***Benefits:***

PDF of k-th detected event time

PDF of sum of k events time

- ***Drawbacks:***

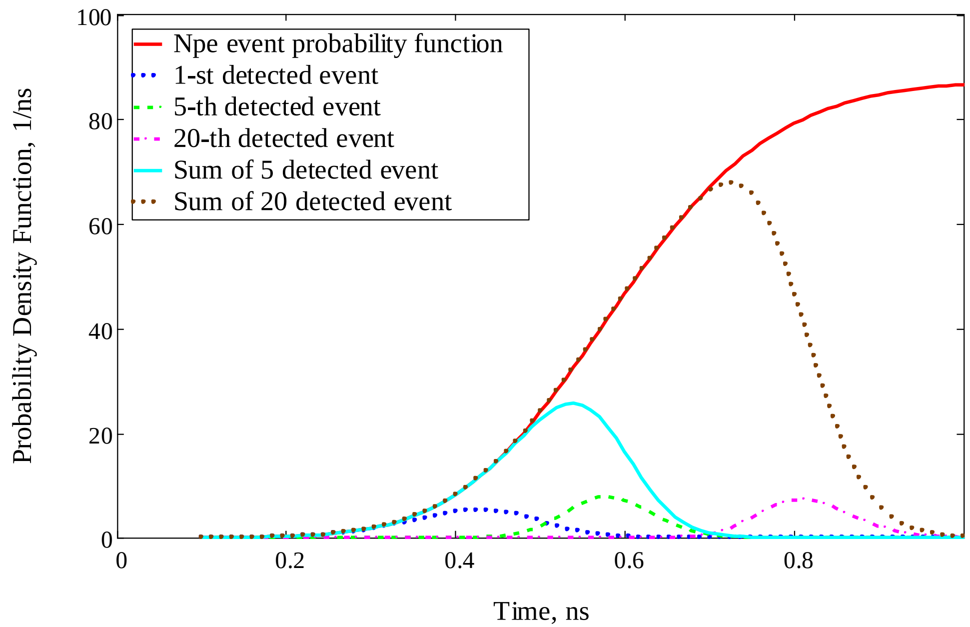
Var[Tk] has to be calculated

Tk not associated with output

=> No information on SER(t)

=> Add crosstalk to every k-th

=> Filtering & composing back



$$PDF(T_{(k):n}) = \sum_{k=1}^{n} PDF(T_{(k):m}) \cdot \binom{n}{k} \cdot Pct^{n-k}(1-Pct)^{k}$$

Simulations & Results

Most popular & demanded case study: LYSO+MPPC

- **LSO & MPPC parameters**

LYSO: 0.09 ns rise, 44 ns decay; 9% resolution

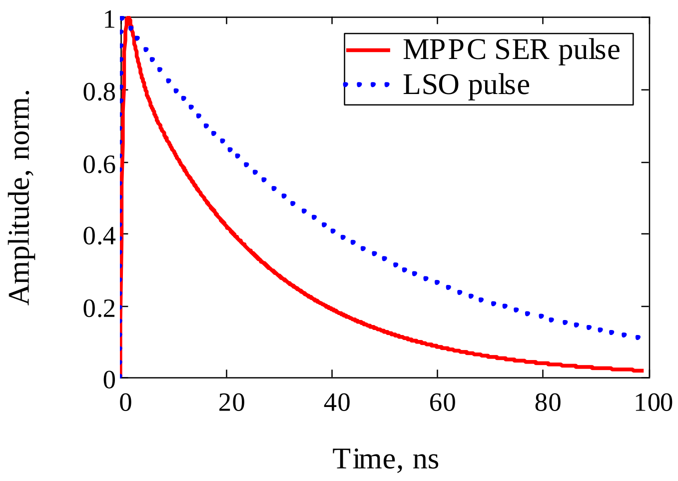
MPPC: Npe=3900, ENFgain=1.015, Pct=0.14; SPTR=0.124 ns, Vnoise=0.32 mV

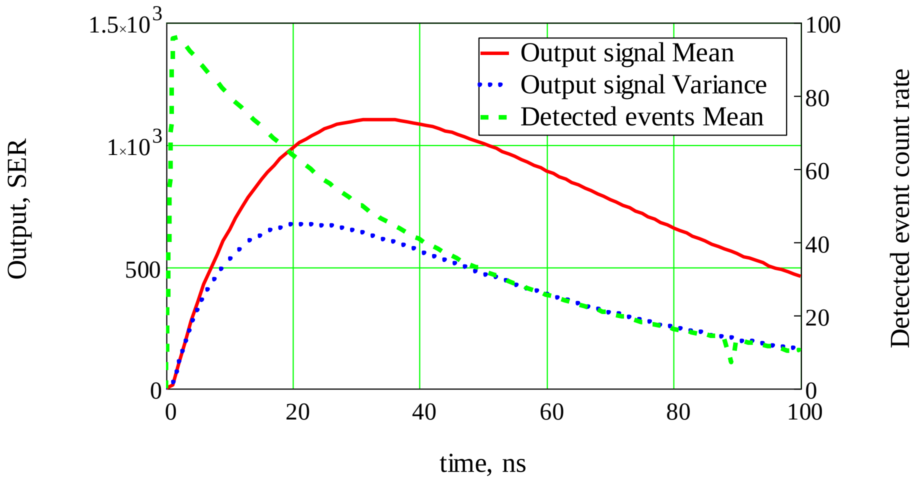
S. Seifert et al, “A Comprehensive Mode to Predict the Timing Resolution ”, TNS, 2012.

- **MPPC SER pulse shape – analytical expression** (~ 1 ns rise, ~ 25 ns decay)

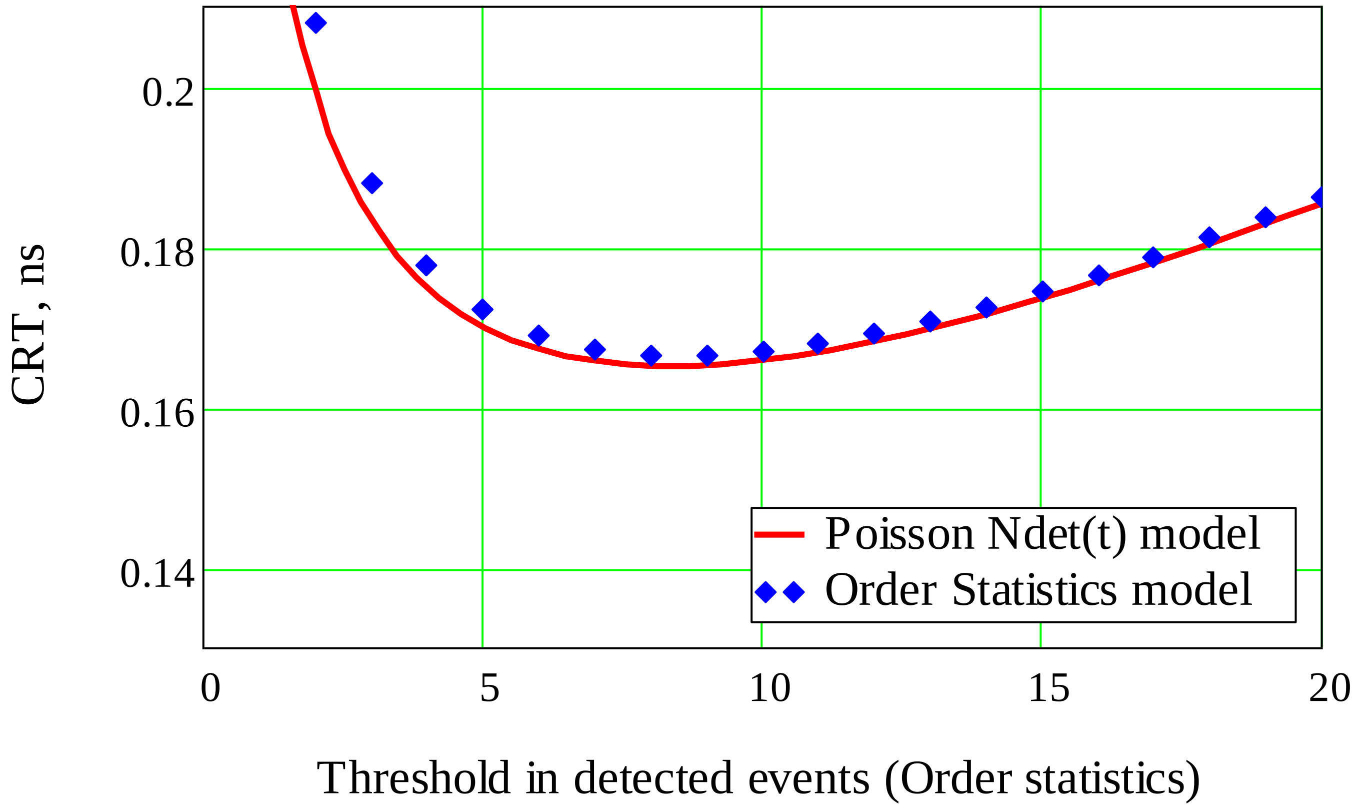
D. Marano et al, “Silicon Photomultipliers Electrical Model: Extensive Analytical Analysis” TNS 2014

Results on Time resolution is given in CRT FWHM = 2.35*√2*σ

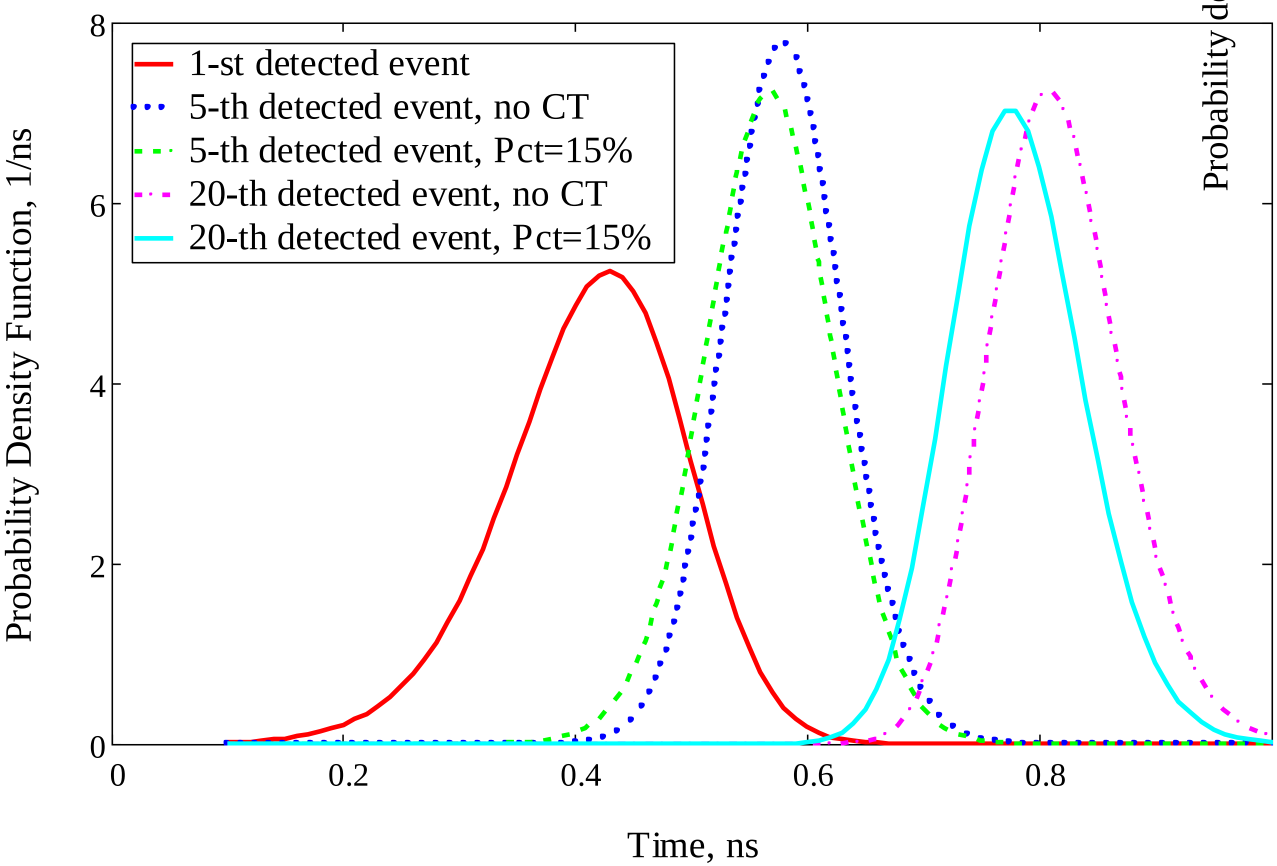


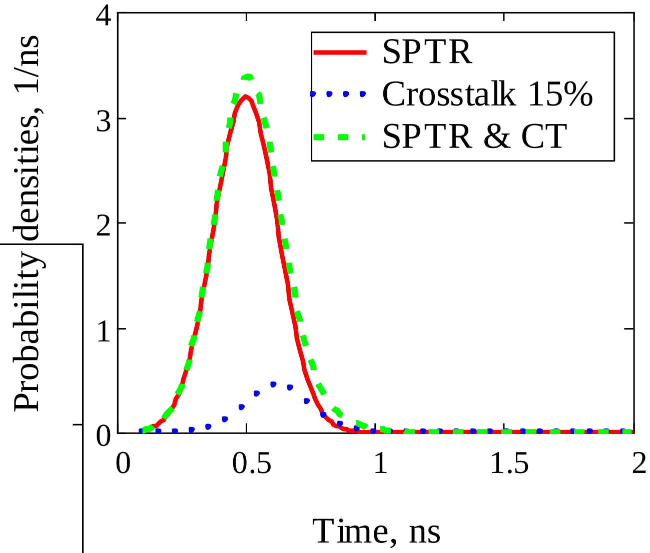


CRT of K-th event (order statistics) vs. Npe(t) (classical)

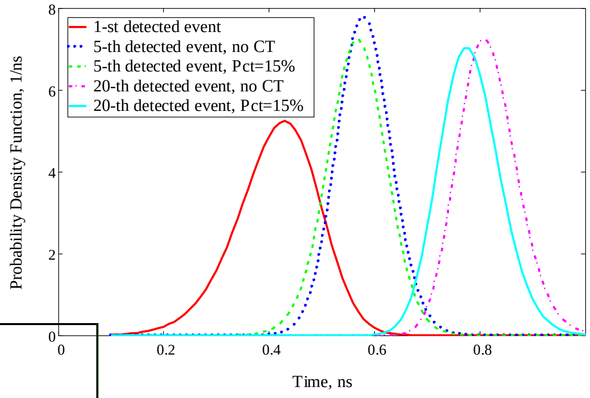


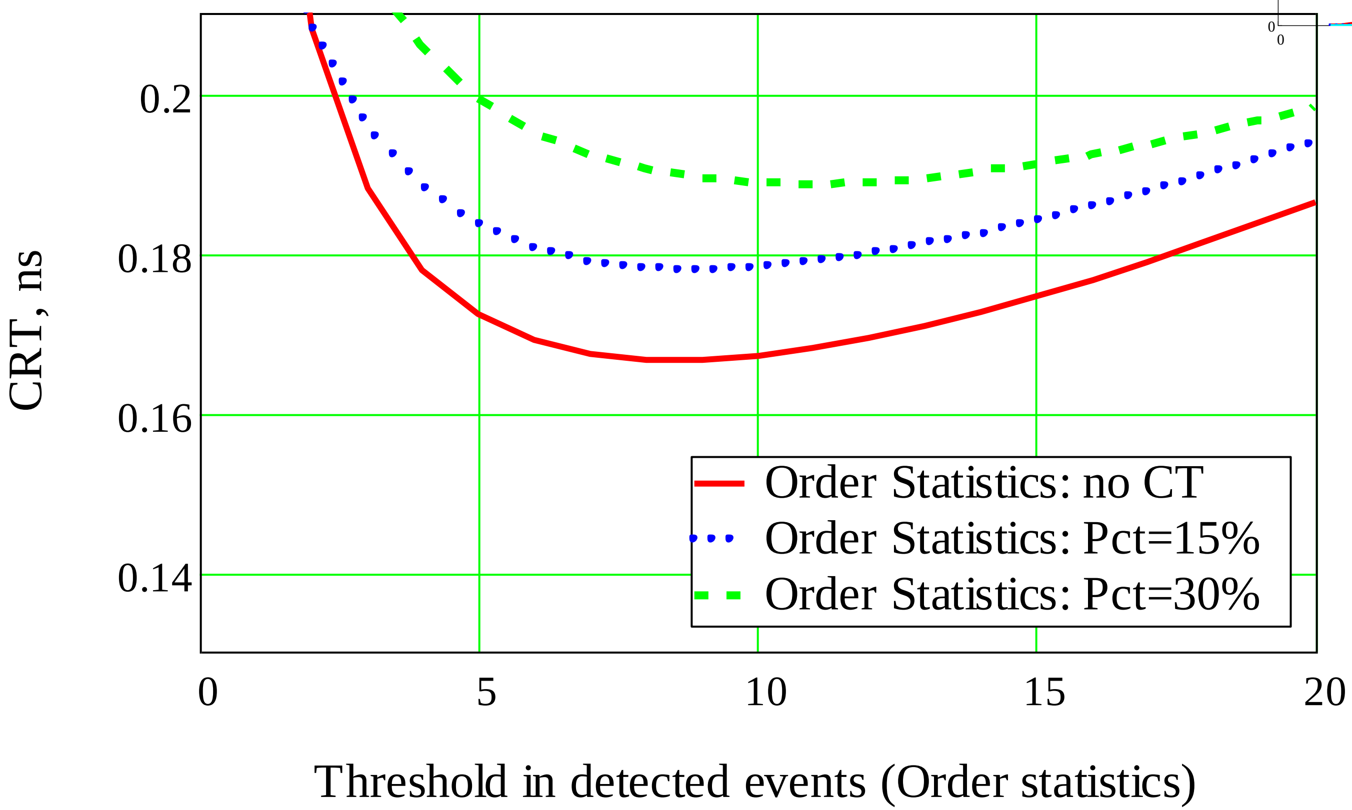
Crosstalk impact on k-th detection time



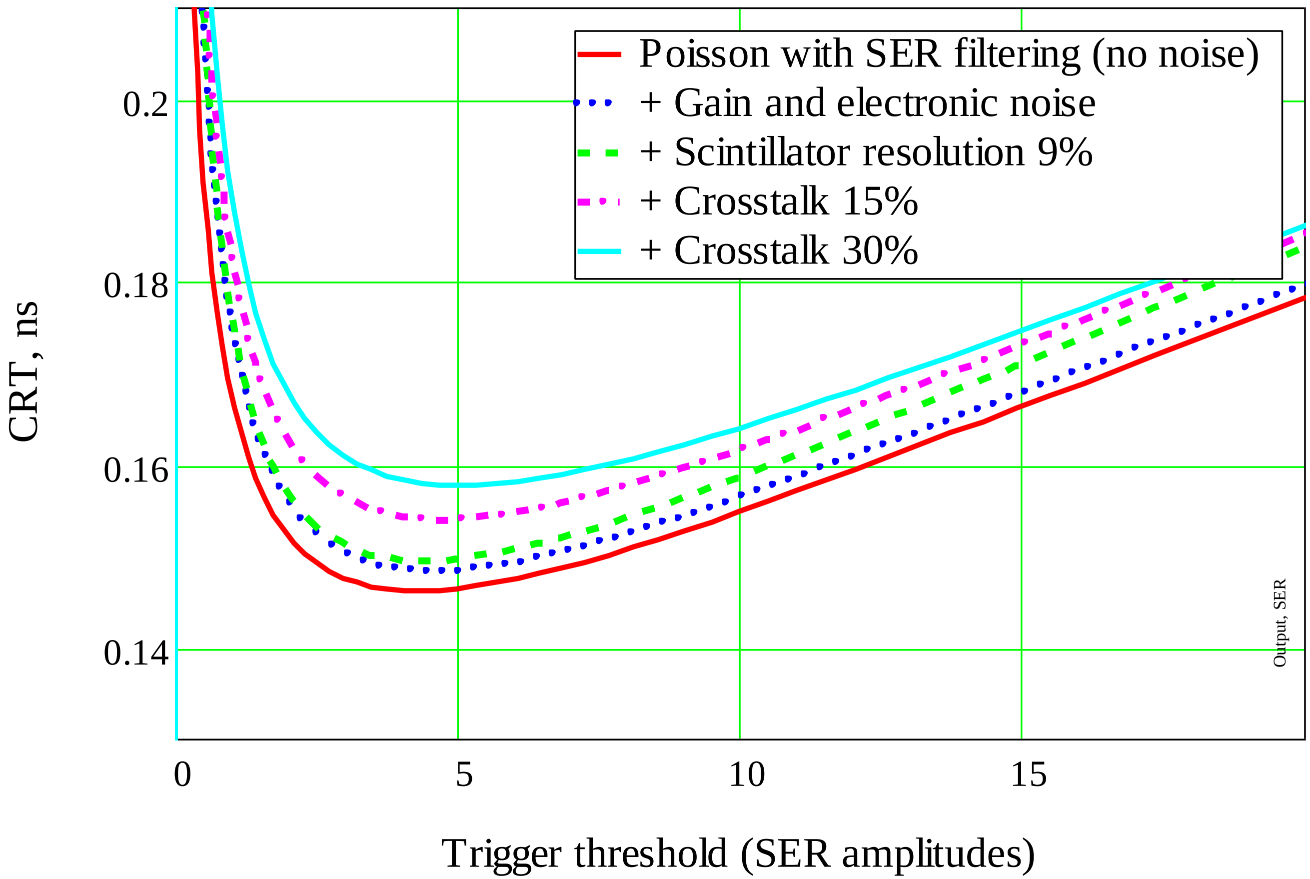


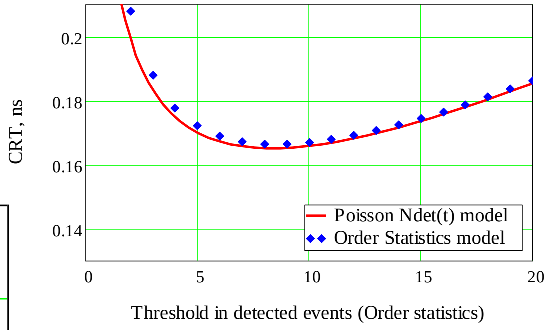
Crosstalk impact on TR (order statistics)



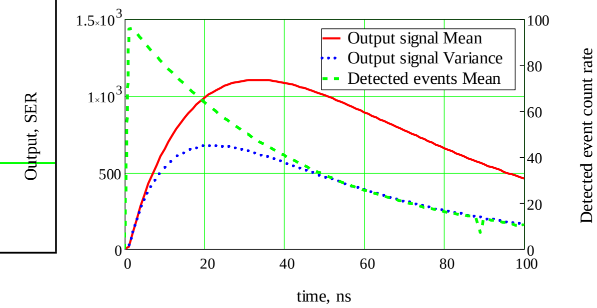


Classical approach with all contributions

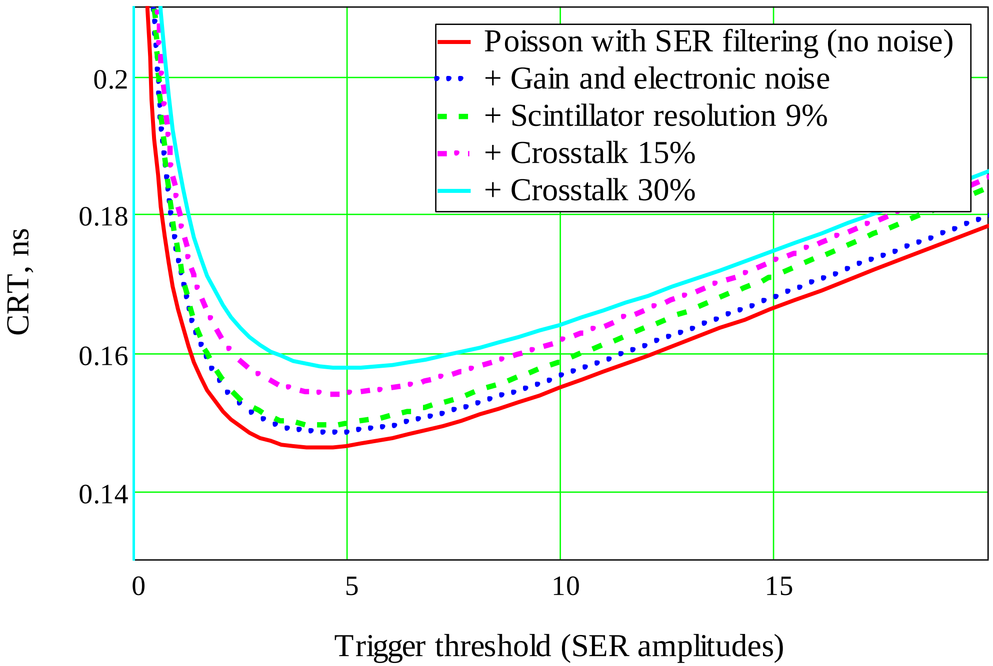


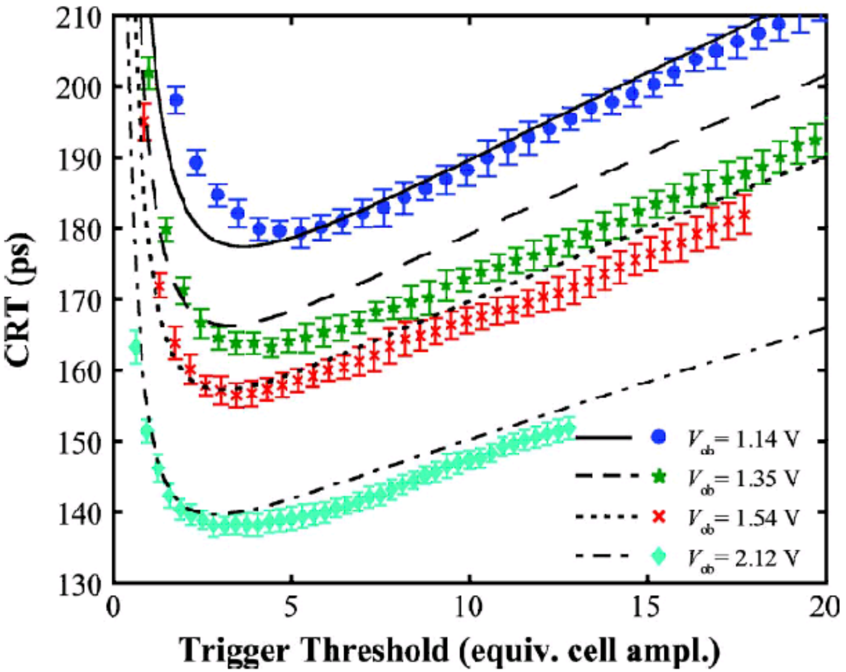


Filtering reduces Variance !

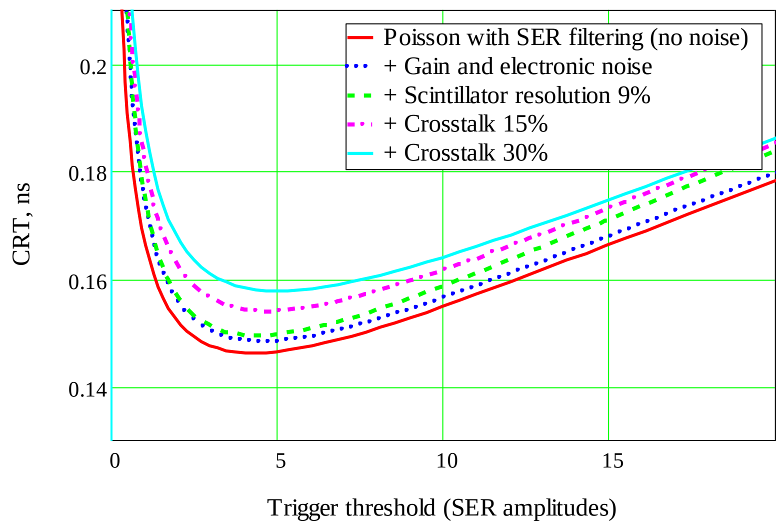


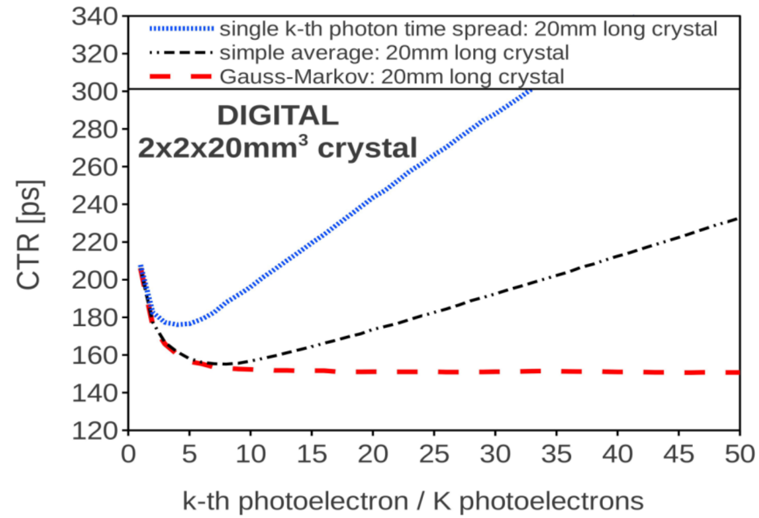
Reference model & experiment





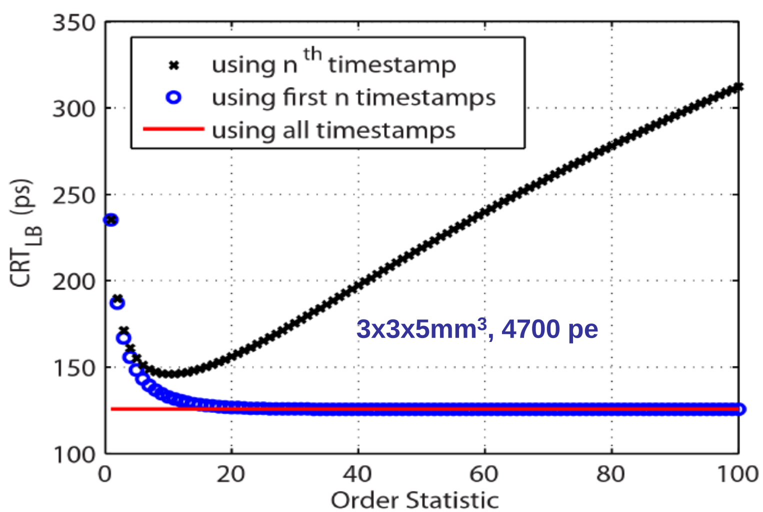
More comparisons



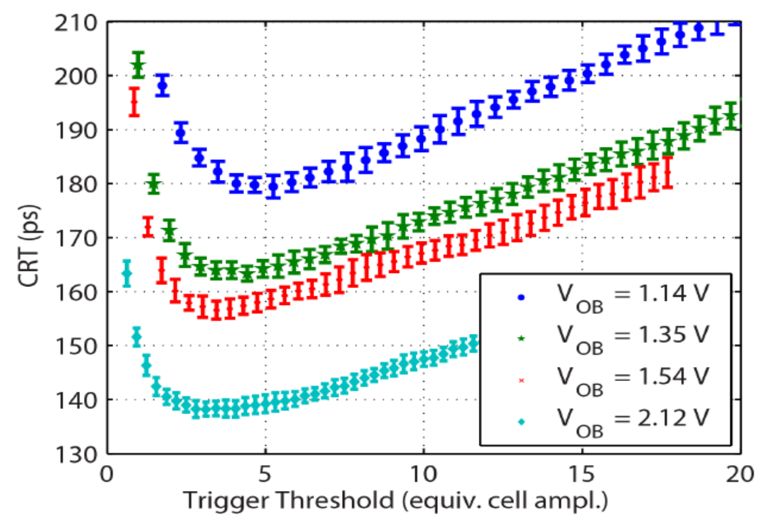


P. Lecoq, E. Auffray, S. Gundacker, CERN, 2014.

Calculated lower bound



Measured CRT



Dennis R. Schaart Prospects for sub-100 picosecond TOF-PET Fast Timing Workshop, LPC Clermont-Ferrand, 12-Mar-2014

Summary

- Classical approach to analytical modelling of TR is powerful and efficient way to account key SiPM factors including crosstalk (and afterpulsing, if required) using advanced “cascaded” technique
- Orders statistics approach is hardly applicable beyond k-th event level to model dSiPM /MD-SiPM as it requires back composition of $\sum k$ into joint distribution
- Simplified ENF-based approach could be useful at fast SER wrt detection event time distribution
- More attention should be paid to clarify applicability of Cramer-Rao lower bound estimate to analog SiPM with correlated events and filtered output.